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Self-affine roughness of a crack front in heterogeneous media

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The long-range elastic model, which is believed to describe the evolution of a self-affine rough crack front, is analyzed to linear and nonlinear orders. It is shown that the nonlinear terms, while important in changing the front dynamics, do not change the scaling exponent which characterizes the roughness of the front. The scaling exponent thus predicted by the model is much smaller than the one observed experimentally. The inevitable conclusion is that the gap between the results of experiments and the model that is supposed to describe them is too large and some new physics has to be invoked for another model.

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The self-affine roughness of a crack front propagating under a tensile load in a randomly heterogeneous system is a well-studied issue, both experimentally and theoretically. Experimentally one measures the position $h(x,t)$ of the crack front, where h is the position of the front as a function of the spanwise coordinate x at time t, and finds that this is a self-affine function whose roughness is characterized by a scaling exponent ζ [defined below in Eq. (4)] in the range of 0.5–0.65 [1,2]. Theoretically there appears to be a consensus that the appropriate model for such dynamical roughening is a long-range elastic string close to its depinning threshold. This model is defined by the equation of motion for a front $h(x,t)$, which is allowed to move only forward due to the irreversibility of the fracture process [3]

$$\frac{\partial h(x,t)}{\partial t} = G^{(0)}[1+2I_1] - \Gamma(x,h), \quad \text{for } \frac{\partial h(x,t)}{\partial t} > 0,$$

$$I_1 \equiv \frac{1}{2\pi}\int dx' \frac{h(x',t)-h(x,t)}{(x'-x)^2}. \qquad (1)$$

Here and below the integral is meant in the Cauchy principal value sense. The right-hand side of Eq. (1) is the difference between the local driving force (below referred to as G, related physically to the energy release rate driving the crack [4]) and $\Gamma(x,h)$ which is a random quenched noise (representing the random material fracture energy [4]). $G^{(0)}$ is the control parameter that represents the energy release rate of a straight front. The integral term stands for the long-range restoring forces stemming from bulk elastic degrees of freedom. The correspondence between the theoretical model and the experimental findings remained, however, unclear, since the best numerical studies of the resulting self-affine graph $h(x)$ of this model came up with a roughness exponent $\zeta = 0.388 \pm 0.002$ [5], clearly outside the range of error of the experimental measurements. This apparent difficulty led to a number of interesting studies, insisting that the model is basically right and that the result concerning the scaling exponent is not final. Thus, for example, in [6] the authors analyzed Eq. (1) using a functional renormalization group. They have calculated the scaling exponent ζ to one- and two-loop orders in ϵ, where $\epsilon = 2-d$. To one-loop order the result is $\zeta = \epsilon/3$, predicting $\zeta = 1/3$ at $d=1$, deviating considerably from the best numerical estimate. To two-loop order the prediction increases to about 0.466, leading to a statement that the model probably describes properly the experimental findings. Unfortunately, it is well known that the ϵ expansion is often an asymptotic series [7], sometimes providing a better estimate of the exponents at first order than at second order. For all that one knows the third loop order may bring the exponent down, maybe even below the first loop order. Another approach was advocated in Refs. [8,9] where it was proposed that the discrepancy between model and experiment may be assigned to the existence of nonlinear contributions to Eq. (1). In [10] the authors derived, in agreement with the results of [11], that to second order in nonlinearity Eq. (1) reads [12]

$$\frac{\partial h(x,t)}{\partial t} = G^{(0)}\left[1+2I_1+I_1^2+2I_2+\frac{1}{4}h'^2\right] - \Gamma(x,h)\left[1+\frac{1}{2}h'^2\right], \quad \text{for } \frac{\partial h(x,t)}{\partial t} > 0,$$

$$I_2 \equiv \frac{1}{4\pi^2}\int\int \frac{[h(x',t)-h(x,t)][h(x'',t)-h(x',t)]}{(x'-x)^2(x''-x')^2}dx'dx'', \qquad (2)$$

where $h' \equiv \partial h/\partial x$. Note that the coefficients are all determined by elasticity theory and are not free. Using a method proposed by Schwartz and Edwards [13], it was concluded that the nonlinear term affects the scaling exponent dramatically, stating that $\zeta \geqslant 0.5$ [9]. On the other hand, in [8] a similar nonlinear equation was analyzed in the framework of the one-loop renormalization group, yielding $\zeta \approx 0.45$. Clearly, the situation warrants some further scrutiny.

In this Rapid Communication we present careful numerical measurements of the scaling exponent of the present model to first and second order in $h(x,t)$. To this aim we simulate the dynamical model (1) with and without the nonlinear terms in Eq. (2). Our final conclusion is that although the second-order terms perturb the solution $h(x,t)$ significantly, they are actually irrelevant for the scaling exponent, which appears unchanged with or without the nonlinear terms. The uneasy conclusion of this analysis is that the model itself may not describe the experiment correctly; a discussion of this conclusion is offered at the end of this paper.

To numerically simulate the model we discretize the spatial variable x and swap temporal changes with discretized steps in the variable $h(x)$. Choosing the basic unit of length to be in the x direction, we present below simulations for $x \in [1,L]$ with $L=2^n$ in the range [2048, 16384]. We used periodic boundary conditions. The discretization of $h(x)$ is chosen in units of 1/7; this seems arbitrary, but since in the depinning limit the velocity is irrelevant, this discretization should not affect the scaling exponents. At step zero the interface is prepared with a random jitter to avoid spurious lattice artifacts. The random quenched noise $\Gamma(x,h)$ is picked at each lattice point from a uniform distribution in the interval [0, 1.5] [14]. Following [15] we simulate the depinning limit by increasing $G^{(0)}$ incrementally from zero, such that the local driving force at the least pinned point overcomes the local fracture energy Γ. This local depinning may trigger additional motion until the interface stops, at which moment $G^{(0)}$ is increased further until the next weakly pinned point gives in. Measurements of the roughness were taken when all the points x moved at least one step after the last increment in $G^{(0)}$.

For the present calculation we employed the rms definition of the roughness: i.e.,

$$w(\ell,L) \equiv \left\langle \frac{1}{\ell}\sum_{x=j}^{j+\ell}\left[h(x) - \frac{1}{\ell}\sum_{x=j}^{j+\ell}h(x)\right]^2\right\rangle_j^{1/2}, \quad \ell \leqslant L. \quad (3)$$

We keep the implicit L dependence in this quantity since it turns out that the roughness exponent is a slowly convergent quantity as a function of L. Indeed, it was convincingly demonstrated in [5] that the numerical value of roughness exponent as measured using the linear model (1) converges in the relation

$$w(\ell,L) \sim \ell^{\zeta} \quad (4)$$

only when L is of the order of 10^6. Not having simulations of this order we resort to finite-size scaling, which was demonstrated [16] to yield reliable exponents also with smaller values of L. The essence of this method is the finite-size scaling assumption written as

$$w(\ell,L) = L^{\zeta}f(l/L); \quad (5)$$

for L very large, there is a range of values of ℓ where $f(l/L) \sim (\ell/L)^{\zeta}$, coalescing with the simple scaling assumption (4). For smaller values of L one seeks the best value of ζ by data collapsing $w(\ell,L)L^{-\zeta}$ onto a universal function $f(\ell/L)$. An example of this procedure is shown in Fig. 1 in which the first-order model had been employed and the results were averaged over 100 realizations for each L. The data collapse appears satisfactory with the choice $\zeta=0.362$. The degree of confidence that this method provides can be demonstrated by the inferior data collapse obtained for the same data with $\zeta=0.4$ in Fig. 2. Our best estimate of the scaling exponent of the model realized to first order is $\zeta=0.365 \pm 0.005$. Note that this exponent is higher than the value $\zeta \approx 0.35$ obtained using Eq. (4) with $L=16384$. This is in agreement with the statements in the literature for the slowness of convergence of the scaling exponent with L [5]. The finite-size scaling analysis improves the situation, even though our estimate still falls a bit short compared to the estimate $\zeta=0.388$ for $L \approx 10^6$ [5]. This difference will not pose a difficulty in assessing the importance of the nonlinear term.

Adding the nonlinear terms, one should first ascertain that they make a significant change in the front dynamics. This is demonstrated in Fig. 3, which compares, for the same initial interface and the same quenched noise $\Gamma(x,h(x))$, the realized interfaces, once with only a linear term and once with the full second-order nonlinear contributions. It is obvious that the nonlinear terms are not negligible; they inflict major changes on the actual graph. The seasoned reader will notice, however, that the scaling exponent is hardly changed. This eyeball conclusion is fully supported by the finite-size scal-

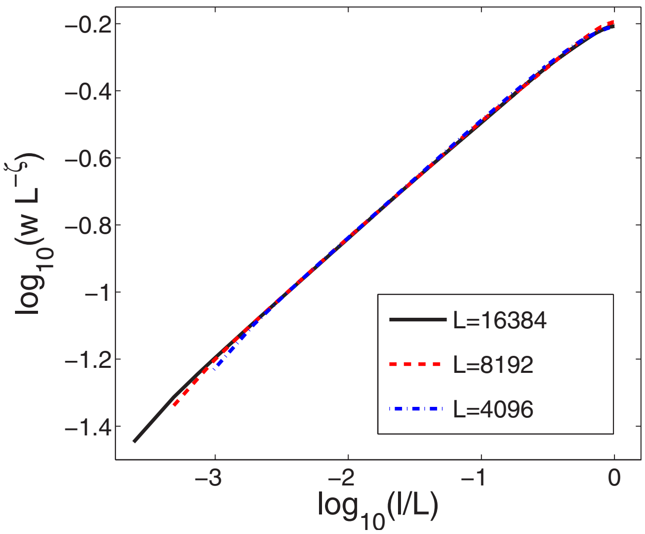

FIG. 1. (Color online) Demonstration of data collapse for the linear model, plotting $\log_{10}(w(\ell,L)L^{-\zeta})$ vs $\log_{10}(\ell/L)$ with $L=2^n$, $n=12$, 13, and 14, using $\zeta=0.362$. The results are obtained by averaging over 100 realizations.

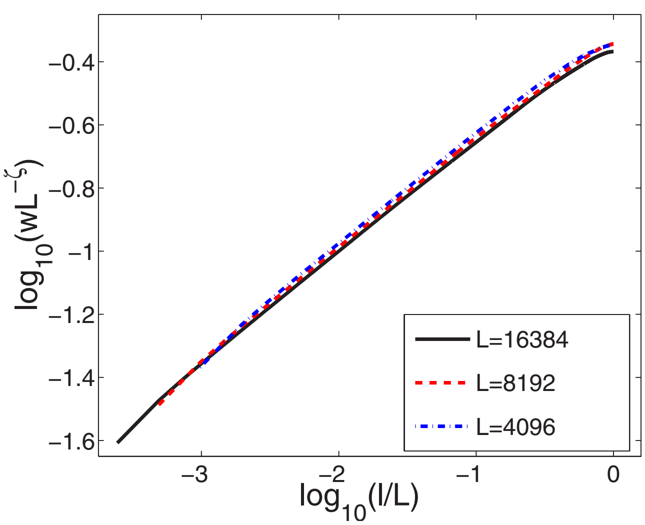

FIG. 2. (Color online) Demonstration of the failure of data collapse for the linear model, plotting $\log_{10}(w(\ell,L)L^{-\zeta})$ vs $\log_{10}(\ell/L)$ with $L=2^n$, $n=12$, 13, and 14, using $\zeta=0.4$. The results are obtained by averaging over 100 realizations.

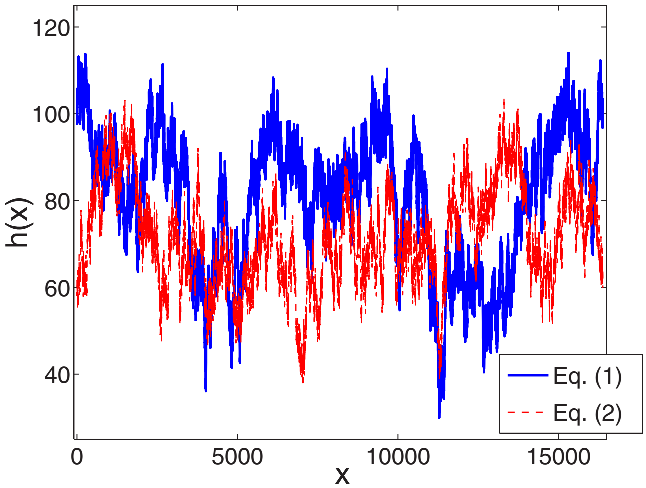

FIG. 3. (Color online) Comparison of the fronts obtained with the linear and the nonlinear model, where the initial condition and the quenched noise are all the same. Note the huge difference in scales between the abscissa and the ordinate.

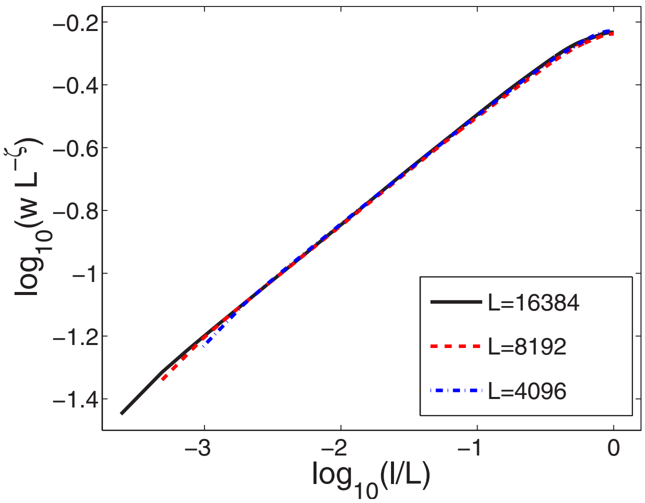

FIG. 4. (Color online) Demonstration of data collapse for the nonlinear model, plotting $\log_{10}(w(\ell,L)L^{-\zeta})$ vs $\log_{10}(\ell/L)$ with $L=2^n$, $n=12$, 13, and 14, using $\zeta=0.362$. The results are obtained by averaging over 100 realizations.

ing analysis that is presented in Fig. 4. The quality of the data collapse is essentially identical in Figs. 1 and 4 *using the same scaling exponent in both cases*. Note that the amplitude of the overall front roughness is reduced in comparison to the linear model; the nonlinearity increases the stiffness of the front. The exponent remains, however, invariant. We thus conclude that the numerical evidence presented here does not support the theoretical propositions of [8,9].

In light of these results we propose that the relation between the experiments and the model must be reassessed. One could argue that our fronts are not large enough to asymptote to a "correct" scaling behavior. To such a claim one must answer that the typical experiments do not have more scales than our simulation. For example, in the fracture experiments of [1] the randomness scale (also known as the correlation length) is determined by the size of sand particles that blast the interface between two slabs of material that are then glued together. This typical scale, which determines the scale of the fracture energy Γ in the present model, is of the order of 50 μm. Crack-front segments up to 50 mm were analyzed, giving at most three orders of scales in theory, but in practice the self-affine scaling was observed in a range that spans about two orders of magnitude. The measured roughness exponent is significantly larger than the values discussed above, even in this limited range of length scales. It is therefore entirely reasonable, in our opinion, to restrict the theoretical analysis of any given model to about the same range of length scales or slightly more, as is done above. Theories invoking asymptotically large system sizes may be interesting, but hardly relevant for such experiments.

Accordingly we may ask what is missing in the relation between theory and experiment. One thing that may be suspicious is the assumption at the background of the derivation of the models (1) and (2)—i.e., that elasticity theory is entirely sufficient to describe the crack-front dynamics. Since elasticity theory predicts the divergence of stress at the crack front [4], realistic materials will almost surely yield plastically or develop additional local damage. Such a change in material properties, precisely where the dynamics is taking place, may very well change the nature of the long-range interactions of the bulk degrees of freedom. How to renormalize such long-range interactions when plasticity or other modes of damage are at play is not known at this point in time. We stress, however, that the gap between the model results and the experimental results indicates that such novel thinking about the theoretical fundamentals may be unavoidable.

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[11] See S. Ramanathan, Ph.D. thesis, Harvard University, 1997 (unpublished) and take the zero-velocity and -frequency limits in Eqs. (3.48), (A.26), and (A.35). Note, however, that a rotation of the "stress intensity factor," from the propagation direction to the normal direction, is missing here and appears properly in [9,10]. Moreover, a proper rotation of the crack-front velocity, back from the normal direction to the propagation direction [resulting in a Kardar-Paris-Zhang- (KPZ-) like term $\sqrt{1+h'^2}$], is included in [9,10].

[12] Note that the term proportional to $\Gamma(x,h)h'^2$ includes a contribution to third order in h due to the implicit dependence of Γ on h. We left this term as is, since it is not obvious how to single out the second-order contributions.

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